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Fair Division of Household Chores

We begin with a set of chores and a set of house members; there are 10 chores (including a "free" week) and 10 house members. Every chore must be completed, however not everyone must complete a chore, and members can complete multiple chores. This assertion leads us to wonder if there is a fair way to divide the chores up amongst the house members so that some people are paying others to do the chores for them, while everyone thinks they are contributing no more than anyone else. Now we need to know what monetary value each person assigns to the chores. This means that person X is indifferent between completing chore A while being paid $Q and simply paying $Q or less to person Y to complete chore A. We assume everyone values their work equal to anyone else's work. After we've collected each member's assignment of monetary values to each of the chores, we see that no two people have placed the same monetary value on any given chore. For example, Andrea believes Composting is worth $10 while Karin only thinks it is worth $3.

We total all the chore values for each member and find the average value. This we will call the Randomly Assigned Average Chore Value (RAACV). This means that each member is indifferent between being randomly assigned a chore and paying their RAACV. We see that each member has a different RAACV, for example Jay's RAACV is $2.05 while Sarah's RAACV is $7.15.

To find a fair division of chores and money, we begin by having each person give their RAACV to the pot. In our model this totals $43.35. Then we have two options of how to assign the chores.

The cheapest labor might possibly produce the lowest quality of labor, so we will call this the **Messy House Method (MHM)** when we assign each chore to the person who is willing to complete it for the least amount of money. For example, Sarah values Vacuuming/Dusting at $0.50 – lower than anyone else, and so she will be assigned this chore.

Then the total cost of the cheapest labor is calculated and divided by the number of players. The total of our model's cheapest labor is $12.50. This is subtracted from the total amount paid and each laborer is paid out of the pot exactly what they offered to complete their chore(s) for. This leaves us with $31.35 in the pot. We again have two choices of how to divide this money amongst the members.

In the **First** distribution each member is given 1/10 of what is left in the pot. In our model this gives a $3.14 payback to each of the members (disregarding the extra $0.05) Now we can calculate each member's percentage contributed from their own perspective by taking their (RAACV - $3.14) divided by their RAACV. For example Anna's RAACV was $2.50, completed Writing Letters, Recycling/Trash, and Watering Plants for $2.00, and so ended up still paying $0.50 into the pot and then receiving $3.14 back. This means that from Anna's perspective, she believes that she contributed ($2.50-$3.14)/$2.50 = -25.40% of her RAACV. This may seem odd, however, because Anna's RAACV was so low, she actually made money from this division of chores. Another example is Jonathan, whose RAACV was $4.10 and he did no chores, so he believes he contributed ($4.10-$3.14)/$4.10 = 23.54% of his RAACV. Using this method each member believes they are paying less than their RAACV.

Another way to distribute the left over money in the pot is to divide it according to each member's RAACV so that each person, from their own perspective, is contributing the same amount after they receive their payback. This is done by determining what percentage of the original pot is left over for distribution. By taking the left over $31.35 in our case divided by the original $43.85, we find that we have 71.49% left over. So each member will receive that percentage of their RAACV back and therefore will end up contributing 100% – 71.49% = 28.51% of their RAACV. For example, Ted's RAACV is $5.15. So $5.15 multiplied by percentage 71.49% so that he is receiving a payback of ($5.15)(0.7149) = $3.68 and so his total perspective percent given is ($5.15-$3.68)/$5.15 = 28.51%. This percentage is the same for every member, which is why we call this distribution **Equal Shares**.

Now we can look at specific house member's perspectives. We see that Hannah, with the **First MHM** sees herself paying 33.89% of her RAACV because she paid $4.75 (her RAACV) and received $3.14 back. However when she looks at Matt she sees that he only paid $4.40 (because this was his RAACV) and he still received $3.14 back. This means that from Hannah's perspective, Matt is only giving 26.53% of the RAACV (this is calculated with Hannah's

RAACV because it is from her perspective) and therefore Matt is getting a better deal than Hannah.

We can analyze each of these four methods in this way by looking at each member's perspective. However, while each person believes they are getting a better deal than if they had been randomly assigned a chore, at least one person can look at another and realize, from their own perspective, that the other person is getting a better deal than they are.

So to get closer to an equal distribution we will now look at a distribution of **No Original Payment**. The other two distributions have each member first giving their RAACV and then getting money back. In the distribution of **NOP** we first total up the cost of money needed to pay for the chores to be done which is $12.50 in our case and then split that cost evenly among each of the members, and then pay whoever is doing the chores the amount for which they specified they would be willing to complete the chore(s). This means each member pays $1.25 minus their payment for whatever work they contributed. For example Jay would contribute $1.25, and then be paid $6.00 for his labor of doing Sweeping/Mopping, Cleaning 1 Lg. Bathroom, and Cleaning 2 Sm Bathrooms. His total contribution is still his RAACV – $1.25 and so he ends up giving ($2.05-$1.25)/$2.05 = 39.02% of his RAACV.

We can look at each member's perceptive with this distribution also, and we see that no one sees that anyone else has to pay less than they themselves; therefore no one can possibly be getting a better deal than they are. So we now speculate that the NOP distribution is a fair distribution of chores and money.

Let's go back to assigning the chores. We saw that we can assign the chores to those who value them the least, but what if assigned the chores to those who valued then the most? The most expensive labor might possibly produce the best quality of labor, so we will call this the **Clean House Method (CHM)** when we assign each chore to the person who is only willing to complete it for the most amount of money. For example, Sarah values Sweeping/Mopping at $23.00 – higher than anyone else, and so she will be assigned this chore.

We can follow the **First** distribution and the **Equal Shares** distribution, but this time, instead of each person getting a payback, there will be an additional amount needed because the total cost of the most expensive labor is more than the total of everyone's RAACVs that is originally totaled in the pot. In our case, this labor costs $95.00 while we only start with the same $43.85 in the pot. This means we need an extra $51.15 which means using the **First** distribution each person will pay an extra $5.12. For example, Krista's RAACV is $3.25, and so she will end up paying ($3.25+$5.12)/$3.25 = 257.38% of total her RAACV. Using the **Equal Shares** distribution, we see that $95.00/$43.85 = 116.65% so everyone ends up paying 116% more of their RAACV for a total of everyone contributing 216.65% in the end.

Looking at the **NOP** distribution we see that each person begins by paying $9.50 because the total amount needed is $95.00. For example, Anna's RAACV is $2.50 and so from her perspective she is paying $9.50/$2.50 = 480.00% of her RAACV.

We can evaluate each of the six different distribution methods by using five different properties.

The Proportionate property: when each member's contribution is at most 1/10 of the total of value of the chores. This is equal to a member's RAACV so a method is Proportionate when they do not exceed giving 100% of their RAACV. Looking at the **MHM**, we see that each member, from their own perspective is giving their RAACV to the pot, and because the chores are being assigned to those who value them the least, there is an excess amount of money in the pot and this is distributed back to each of the players. Therefore every member never gives any more than 100% of their RAACV. In the case of the **NOP MHM** each member is originally giving their equal share of the total needed to pay for the cheapest labor – this is never any more than 100% of their RAACV because everyone's total for all the chores is at least as much as the total needed for the cheapest labor. <u>Therefore the **MHM** is proportionate.</u> Looking at the **CHM**, we see that each member originally gives their RAACV to the pot, however because the chores are being assigned to those who value them the most, there is a lack of money in the pot and this extra money that is needed comes from each one of the members. Therefore every member ends up paying more than 100% of the RAACV. Looking at the **NOP CHM** each member is originally giving their equal share of the total needed to pay for the most expensive labor – this is always more than 100% of their RAACV because everyone's total for all the chores is at most as much as the total needed for the most expensive labor. Therefore the **CHM** is not proportionate.

The Envy Free Property: when we see that every member values the work/money every other member contributed at least as much as they value the work/money they contributed. We can see that in the **NOP MHM** each member paid the same minimum amount of money needed to afford the cheapest labor – and so therefore those who were willing to work for the least amount of payment completed the chores and when they were paid for their labor, they believed they were getting at least as good a deal as everyone else. Those who paid others to do the chores for them valued the chores at a higher price but only had to pay for the cheaper labor. Therefore these people also believed they were getting at least as good a deal as everyone else, and so the **NOP MHM** is Envy Free.

All the other methods as we can see are NOT Envy Free because those paying others to do the chores for them all value the chores at different prices, and so with the **First** and Equal Shares both **MHM** and **CHM** inevitably at least two members who are not doing any work but paying money, will be paying different amounts of money and therefore one will be jealous of the other's lower contribution and this makes the entire method NOT Envy Free. Also with the **NOP CHM**, each member is paying for more than they think the chores are worth and so to those who are not doing any labor and only paying others thinks they are not getting as good of a deal as anyone that is doing a chore and therefore this method is NOT Envy Free. <u>Therefore the **NOP MHM** is the only Envy Free method.</u>

Efficient Property: any other exchange of money or work will increase the contribution of one member and decreases the contribution of another. We see that the **CHM** is not efficient because by reassigning any chore to someone who is willing to complete it for less money, one member can lower their contribution because less money is being spent to pay for the same labor being done while the other member is being paid for the work they are contributing and so their overall contribution remains the same. However the **MHM** is efficient because no person can be paid less to do a chore than who is being paid now. Therefore, any transference of money or labor would be strictly at a loss to one member and at a gain to another. Therefore only the **MHM** is efficient.

Value Equitable Property: each member's monetary value is equal. In the **First** and **Equal Shares** methods both **MHM** and **CHM**, every member is contributing a different amount of money because everyone has a different RAACV. While in the **First MHM**, each player receives $3.14 back, each person still paid a different RAACV to the pot to begin with. Therefore these four methods are NOT Value Equitable. Now we can see that only the **NOP** methods are Value Equitable because everyone initially pays the same amount of money into the pot and then those who are completing chores are paid for their work so they are not contributing any more than any of the other members. This results in everyone's contributed value being the same and therefore the **NOP** distribution **MHM** and **CHM** are Value Equitable.

Share Equitable Property: each member's percent contribution is equal. It is very obvious that both the **Equal Shares MHM** and **CHM** are Share Equitable because this property is the basis of the distribution. Each member is contributing the same percentage of their RAACV as everyone else is contributing. However the rest of the distributions, **First** and **NOP** both **MHM** and **CHM** are not Share Equitable because each member is paying or receiving the same amount of money (minus the cost of the work they contributed) and no one has the same RAACV and so everyone is paying a different percentage from their own perspective.

Our conjecture is that if **Proportionate, Efficient, Envy Free**, and **Value Equitable** then it must be the **NOP MHM.** Let's begin by assuming this unknown method IS **Proportionate, Efficient, Envy Free**, and **Value Equitable**. First we know that a method is only efficient when the chores are assigned to those who value them the least. Otherwise the chore could be given to someone else who valued it less and this would mean that the member paying someone else less money to do the chore is lowering their contribution while no one is increasing their contribution. This would not be efficient. However if the chores are assigned to those who value them the least, then no one can pay another less money for the same labor and then there are no opportunities to decrease the overall contribution from all the players So, any further trades of money and labor would benefit one or more members would have to harm one or more different members. Therefore this distribution of chores is efficient. Thus the only way for a method to be efficient is if the chores are assigned to those who value them the least.

If our method is Proportionate, then we know each member must not contribute any more than their RAACV to the pot. Therefore the total cost of the labor must be less than or equal to any one member's total value for all the chores. Since we have just shown that we must assign the chores to those who value them the least, then we know that no one will pay more than their RAACV to the pot.

If our method is Envy Free, each member must see that they are getting at least a good a deal as everyone one else in the house. This means that no one sees their % contribution as any more than anyone else's % contribution. Since it is possible for two people to both be paying others to do chores for them, we must start by having everyone pay the same amount of money so that these two members will not believe one is contributing less than the other. The only way to ensure that no one is paying more than their RAACV is to total cost of the cheapest labor and divide that cost evenly among the players. This way, any one member is paying at most their RAACV to the pot because the total cost of the labor must be less than or equal to any one member's total value of all the chores. Therefore their share would be less than or equal to their RAACV. Everyone contributing labor will be paid with the money in the pot for their labor and each member will still be contributing the same value of money originally. Because the chores have been assigned to those who value them the least, no other member is willing to complete the chores for any less and therefore no member is envious of another's contribution.

When a method is Value Equitable, this means that each member is contributing the same value of money and/or work. This means that everyone must pay the same original amount of money into the pot and those contributing work will be paid for their labor so that their contribution is no more than anyone else's.

This brings us to the conclusion that our unknown method which is **Proportionate, Efficient, Envy Free**, and **Value Equitable** begins by assigning the chores to those who value them the least. Then the money needed to pay for this cheap labor is totaled and the cost is then split evenly among each of the members. Then whoever is doing the chores is paid the amount for which they specified they would be willing to complete the chore(s). This means each member pays their equal share of the total cost minus their payment for whatever work they contributed. We see that this is in fact the exact definition of the **No Original Payment Messy House Method.**